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Generalized Kato Decomposition and Essential Spectra

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Abstract Let **R** denote any of the following classes: upper (lower) semi-Fredholm operators, Fredholm operators, upper (lower) semi-Weyl operators, Weyl operators, upper (lower) semi-Browder operators, Browder operators. For a bounded linear operator T on a Banach space X we prove that $T = T_M \oplus T_N$ with $T_M \in \mathbf{R}$ and T_N quasinilpotent (nilpotent) if and only if T admits a generalized Kato decomposition (T is of Kato type) and 0 is not an interior point of the corresponding spectrum $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathbf{R}\}$. Moreover, we prove that if $T - \lambda_0$ admits a generalized Kato decomposition, then $\sigma_{\mathbf{R}}(T)$ does not cluster at λ_0 if and only if λ_0 is not an interior point of $\sigma_{\mathbf{R}}(T)$. As a consequence we get several results on cluster points of essential spectra. In that way we extend some results regarding the approximate point spectrum and the surjective spectrum given by Aiena and Rosas (J. Math. Anal. Appl. 279:180–188, 2003), as well as results given by Jiang and Zhong (J. Math. Anal. Appl. 356:322–327, 2009) to the cases of essential spectra.

Keywords Banach space · Generalized Kato decomposition · Generalized Kato spectrum · Upper and lower semi-Fredholm operators · Semi-Weyl and semi-Browder operators

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1 Introduction

Given a Banach space X and an operator $T \in L(X)$, T is said to be Drazin invertible, if there exists $S \in L(X)$ and some $m \in \mathbb{N}$ such that

$$T^m = T^m ST, \quad STS = S, \quad ST = TS.$$

It is a classical result that necessary and sufficient for T to be Drazin invertible is $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 nilpotent; see [18,24]. Drazin invertible operators and Fredholm operators were generalized to B-Fredholm operators by M. Berkani [3]. According to [3, Theorem 2.7] T is B-Fredholm if and only if $T = T_1 \oplus T_2$ with T_1 Fredholm and T_2 nilpotent. For more details about the B-Fredholm operators we refer the reader to [3–6].

It is said that $T \in L(X)$ admits a Kato decomposition or T is of Kato type if there exist two closed T-invariant subspaces M and N such that $X = M \oplus N$, T_M is Kato and T_N is nilpotent. T. Kato proved in [17] that semi-Fredholm operators admit a Kato decomposition with N finite-dimensional. It is not difficult to see that every B-Fredholm operator admits a Kato decomposition. In [22] Labrouse introduced and studied quasi-Fredholm operators in the context of a Hilbert space. He showed that quasi-Fredholm operators are precisely those admitting a Kato decomposition.

If we require that T_N is quasinilpotent instead of nilpotent in the definition of the Kato decomposition, then it leads us to the generalized Kato decomposition. Operators that admit a generalized Kato decomposition were firstly studied by M. Mbekhta in [26] and he called them pseudo-Fredholm operators.

J. Koliha extended the concept of Drazin invertibility and introduced generalized Drazin invertible operators [19]. According to his work, an operator $T \in L(X)$ is generalized Drazin invertible if and only if 0 is not an accumulation point of the spectrum of T, and it is exactly when $T = T_1 \oplus T_2$ with T_1 invertible and T_2 quasinilpotent. The class of generalized Drazin invertible operators were extended [12] in a way that it was considered the class of operators that can be represented as the direct sum of a bounded below (surjective) operator and a quasinilpotent operator. Very recently, pseudo B-Fredholm and pseudo B-Weyl operators were defined in a sense that T is pseudo B-Fredholm (resp. pseudo B-Weyl) if $T = T_1 \oplus T_2$, where T_1 is Fredholm (resp. Weyl) and T_2 is quasinilpotent [7,29].

In accordance with these observations it is natural to study various types of the direct sums. Namely, let $\mathbf{R}$ denote any of the following classes: upper (lower) semi-Fredholm operators, Fredholm operators, upper (lower) semi-Weyl operators, Weyl operators, upper (lower) semi-Browder operators, Browder operators, bounded below operators, surjective operators, invertible operators. The main objective of this article is to provide necessary and sufficient conditions for an operator $T \in L(X)$ to be the direct sum of an operator $T_1 \in \mathbf{R}$ and a quasinilpotent (nilpotent) operator T_2.

In Sect. 2 we set up terminology and recall necessary facts. Our main results are established in Sects. 3 and 4. Given an operator $T \in L(X)$, X is a Banach space, we prove that $T = T_M \oplus T_N$ with $T_M \in \mathbf{R}$ and T_N quasinilpotent (T_N nilpotent) if and only if T admits a generalized Kato decomposition (T is of Kato type) and 0 is not an interior point of $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathbf{R}\}$ (see Theorems 3.4, 3.6, 3.7, 3.9, 4.1 below). Moreover, we prove that if $T - \lambda_0$ admits a generalized Kato decomposition, then $\sigma_{\mathbf{R}}(T)$ does not cluster at λ_0 if and only if λ_0 is not an interior point of $\sigma_{\mathbf{R}}(T)$. In that way we extend to the cases of essential spectra the result given by Jiang and Zhong [13, Theorem 3.5 and Theorem 3.9] where they show that if $T - \lambda_0$ admits a GKD, λ_0 is not an accumulation point of its approximate point (surjective) spectrum if and only if λ_0 is not an interior point of the approximate point (surjective) spectrum of T (see Corollary 3.5 below). Also, we extend to the cases of essential spectra, as well the approximate point and surjective spectrum, the result of Aiena and Rosas [2, Theorem 2.9] (the result of Jiang and Zhong [13, Theorem 3.8]) which is equivalent to the following assertion: if 0 is a boundary point of $\sigma(T)$, then T is of Kato type (T admits a GKD) if and only if T is Drazin (generalized Drazin) invertible, that is $T = T_M \oplus T_N$ where $0 \notin \sigma(T_M)$ and T_N nilpotent (T_N quasinilpotent) (see Corollary 4.2 (Corollary 3.12) below).

Section 5 contains some applications. We prove that every boundary point of $\sigma_{\mathbf{R}}(T)$, where $\mathbf{R}$ is any of the classes mentioned above, which is also an accumulation point of $\sigma_{\mathbf{R}}(T)$ belongs to the generalized Kato spectrum. In particular, if $\mathbf{R}$ is the class of invertible operators we obtain [13, Corollary 3.6]. If $T \in L(X)$, let $\sigma_{\mathbf{gDR}}(T)$ be the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ can not be represented as the direct sum of an operator from the class $\mathbf{R}$ and a quasinilpotent operator. We show that the connected hull of the spectrum $\sigma_{\mathbf{gDR}}(T)$ coincide with the connected hull of the generalized Kato spectrum for every class $\mathbf{R}$. In particular, the connected hulls of the generalized Drazin spectrum and the generalized Kato spectrum are equal and as a consequence of this fact we get Theorem 3 in [14]. Moreover, the connected hulls of the B-Fredholm, B-Weyl, Drazin and of the Kato type spectrum are equal. Also, from the condition $\sigma_{\mathbf{R}}(T) = \partial\sigma_{\mathbf{R}}(T) = \operatorname{acc}\sigma_{\mathbf{R}}(T)$ we derive $\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{R}}(T) = \sigma_{\mathbf{gDR}}(T)$ for every aforementioned class $\mathbf{R}$.

2 Preliminaries

Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ denote the set of all complex numbers. Let X be an infinite dimensional Banach space and let $L(X)$ be the Banach algebra of all bounded linear operators acting on X. The group of all invertible operators is denoted by $L(X)^{-1}$. Given $T \in L(X)$, we denote by $N(T)$, $R(T)$ and $\sigma(T)$, the *kernel*, the *range* and the *spectrum* of T, respectively. In addition, $\alpha(T)$ and $\beta(T)$ will stand for *nullity* and *defect* of T. The space of bounded linear functionals on X is denoted by X'. If $K \subset \mathbb{C}$, then ∂K is the boundary of K, $\operatorname{acc} K$ is the set of accumulation points of K, $\operatorname{iso} K = K \backslash \operatorname{acc} K$ and $\operatorname{int} K$ is the set of interior points of K. For $\lambda_0 \in \mathbb{C}$, the open disc, centered at λ_0 with radius ϵ in $\mathbb{C}$, is denoted by $D(\lambda_0, \epsilon)$.

Recall that T is said to be *nilpotent* when $T^n = 0$ for some $n \in \mathbb{N}$, while T is *quasinilpotent* if $\|T^n\|^{1/n} \to 0$, that is $T - \lambda \in L(X)^{-1}$ for all complex $\lambda \neq 0$. An operator $T \in L(X)$ is *bounded below* if there exists some $c > 0$ such that $c\|x\| \leq$

$\|Tx\|$ for every $x \in X$. Let $\mathcal{M}(X)$ denote the set of all bounded below operators, and let $\mathcal{Q}(X)$ denote the set of all surjective operators. The approximate point spectrum of $T \in L(X)$ is defined by

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$$

and the surjective spectrum is defined by

$$\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}.$$

An operator $T \in L(X)$ is *Kato* if $R(T)$ is closed and $N(T) \subset R(T^n)$, $n \in \mathbb{N}_0$. If $R(T)$ is closed and $\alpha(T) < \infty$, then $T \in L(X)$ is said to be *upper semi-Fredholm*. An operator $T \in L(X)$ is *lower semi-Fredholm* if $\beta(T) < \infty$. The set of upper semi-Fredholm operators (lower semi-Fredholm operators) is denoted by $\Phi_+(X)$ ($\Phi_-(X)$). If T is upper or lower semi-Fredholm operator then the index of T is defined as $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator T is *Fredholm* if both $\alpha(T)$ and $\beta(T)$ are finite. We will denote by $\Phi(X)$ the set of Fredholm operators. The sets of *upper semi-Weyl*, *lower semi Weyl* and *Weyl* operators are defined by $\mathcal{W}_+(X) = \{T \in \Phi_+(X) : \text{ind}(T) \leq 0\}$, $\mathcal{W}_-(X) = \{T \in \Phi_-(X) : \text{ind}(T) \geq 0\}$ and $\mathcal{W}(X) = \{T \in \Phi(X) : \text{ind}(T) = 0\}$, respectively. B-Fredholm and B-Weyl operators were introduced and studied by M. Berkani [3–5]. An operator $T \in L(X)$ is said to be B-Fredholm (B-Weyl) if there is $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the restriction $T_n \in L(R(T^n))$ of T to $R(T^n)$ is Fredholm (Weyl). The B-Fredholm and the B-Weyl spectrum of T are defined by

$$\sigma_{B\Phi}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Fredholm}\},$$
$$\sigma_{B\mathcal{W}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl}\}, \text{ respectively.}$$

Recall that $T \in L(X)$ is said to be *Riesz operator*, if $T - \lambda \in \Phi(X)$ for every non-zero $\lambda \in \mathbb{C}$.

The *ascent* of T is defined as $\text{asc}(T) = \inf\{n \in \mathbb{N}_0 : N(T^n) = N(T^{n+1})\}$, and *descent* of T is defined as $\text{dsc}(T) = \inf\{n \in \mathbb{N}_0 : R(T^n) = R(T^{n+1})\}$, where the infimum over the empty set is taken to be infinity. An operator $T \in L(X)$ is *upper semi-Browder* if T is upper semi-Fredholm and $\text{asc}(T) < \infty$. If $T \in L(X)$ is lower semi-Fredholm and $\text{dsc}(T) < \infty$, then T is *lower semi-Browder*. Let $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) denote the set of all upper (lower) semi-Browder operators. The set of Browder operators is defined by $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

If M is a subspace of X such that $T(M) \subset M$, $T \in L(X)$, it is said that M is *T-invariant*. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in M$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_M \oplus T_N$ and say that T is the *direct sum* of T_M and T_N.

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated as GKD, if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent. A relevant case is obtained if we assume that T_N is nilpotent. In this case T is said to be of *Kato type*. An operator is said to be *essentially Kato* if it admits a GKD (M, N) such that N is finite-dimensional. If T is essentially Kato then T_N is nilpotent, since every quasinilpotent operator on a finite dimensional space is nilpotent. The classes $\Phi_+(X)$, $\Phi_-(X)$, $\Phi(X)$, $\mathcal{B}_+(X)$, $\mathcal{B}_-(X)$, $\mathcal{B}(X)$, $\mathcal{W}_+(X)$, $\mathcal{W}_-(X)$ and $\mathcal{W}(X)$ belong to the class of essentially Kato operators [27, Theorem 16.21]. For $T \in L(X)$, *the Kato spectrum*, *the essentially Kato spectrum*, *the Kato type spectrum* and *the generalized Kato spectrum* are defined by

$$\begin{aligned}
\sigma_K(T) &= \{\lambda \in \mathbb{C} : T - \lambda \ \text{ is not Kato}\},\\
\sigma_{eK}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \ \text{ is not essentially Kato}\},\\
\sigma_{Kt}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \ \text{ is not of Kato type}\},\\
\sigma_{gK}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \ \text{ does not admit a GKD}\},
\end{aligned}$$

respectively. Clearly,

$$\sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_K(T) \subset \sigma_{ap}(T) \cap \sigma_{su}(T). \tag{2.1}$$

The *quasinilpotent part* $H_0(T)$ of an operator $T \in L(X)$ is defined by

$$H_0(T) = \left\{ x \in X : \lim_{n\to+\infty} \|T^n x\|^{1/n} = 0 \right\}.$$

It is easy to verify that $H_0(T) = \{0\}$ if T is bounded below. An operator $T \in L(X)$ is quasinilpotent if and only if $H_0(T) = X$ [1, Theorem 1.68].

The *analytical core* of T, denoted by $K(T)$, is the set of all $x \in X$ for which there exist $c > 0$ and a sequence $(x_n)_n$ in X satisfying

$$Tx_1 = x, \quad Tx_{n+1} = x_n \ \text{ for all } \ n \in \mathbb{N}, \quad \|x_n\| \le c^n \|x\| \ \text{ for all } \ n \in \mathbb{N}.$$

If T is surjective, then $K(T) = X$ [1, Theorem 1.22].

An operator $T \in L(X)$ is said to be *generalized Drazin invertible*, if there exists $B \in L(X)$ such that

$$TB = BT, \quad BTB = B, \quad TBT - T \ \text{ is quasinilpotent.}$$

The generalized Drazin spectrum of $T \in L(X)$ is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not generalized Drazin invertible}\}.$$

The equivalent conditions to the existence of generalized Drazin inverse of a bounded operator are collected in the following theorem.

Theorem 2.1 (see [8,19,20,25,28]) *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is generalized Drazin invertible;*
(ii) *There exists a bounded projection P on X which commutes with T such that $T + P$ is invertible and TP is quasinilpotent;*

(iii) $0 \notin \operatorname{acc} \sigma(T)$;
(iv) *There is a bounded projection P on X such that $R(P) = H_0(T)$ and $N(T) = K(T)$;*
(v) *There exists $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasinilpotent;*
(vi) *$X = K(T) \oplus H_0(T)$ with at least one of the component spaces closed.*

For a subspace M of X its annihilator $M^\perp$ is defined by

$$M^\perp = \{f \in X' : f(x) = 0 \text{ for all } x \in M\}.$$

Recall that if M is closed, then

$$\dim M^\perp = \operatorname{codim} M. \tag{2.2}$$

Let M and L be two subspaces of X and let

$$\delta(M, L) = \sup\{dist(u, L) : u \in M,\ \|u\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, L) = 0$ for any subspace L. The *gap* between M and L is defined by

$$\hat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}.$$

It is known that [27, corollary 10.10]

$$\hat{\delta}(M, L) < 1 \Longrightarrow \dim M = \dim L. \tag{2.3}$$

If M and L are closed subspaces of X, then [27, Theorem 10.8]

$$\hat{\delta}(M^\perp, L^\perp) = \hat{\delta}(M, L). \tag{2.4}$$

Therefore, for closed subspaces M and L of X, according to (2.2), (2.3) and (2.4), there is implication

$$\hat{\delta}(M, L) < 1 \Longrightarrow \operatorname{codim} M = \operatorname{codim} L. \tag{2.5}$$

We use the following notation.

$\mathbf{R}_1 = \Phi_+(X)$	$\mathbf{R}_2 = \Phi_-(X)$	$\mathbf{R}_3 = \Phi(X)$
$\mathbf{R}_4 = \mathcal{W}_+(X)$	$\mathbf{R}_5 = \mathcal{W}_-(X)$	$\mathbf{R}_6 = \mathcal{W}(X)$
$\mathbf{R}_7 = \mathcal{B}_+(X)$	$\mathbf{R}_8 = \mathcal{B}_-(X)$	$\mathbf{R}_9 = \mathcal{B}(X)$
$\mathbf{R}_{10} = \mathcal{M}(X)$	$\mathbf{R}_{11} = \mathcal{Q}(X)$	$\mathbf{R}_{12} = L(X)^{-1}$

The sets $\mathbf{R}_i$, $1 \leq i \leq 12$, are open in $L(X)$ and contain $L(X)^{-1}$ (for the openness of the set of upper (lower) semi-Browder operators see [21, Satz 4]). The spectra $\sigma_{\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathbf{R}_i\}$, $1 \leq i \leq 12$, are non-empty and compact subsets of $\mathbb{C}$. We write $\sigma_{\mathbf{R}_1}(T) = \sigma_{\Phi_+}(T)$, $\sigma_{\mathbf{R}_2}(T) = \sigma_{\Phi_-}(T)$, etc, and $\rho_{\Phi_+}(T) = \mathbb{C}\backslash\sigma_{\Phi_+}(T)$, $\rho_{\Phi_-}(T) = \mathbb{C}\backslash\sigma_{\Phi_-}(T)$, etc. In particular, $\sigma_{\mathbf{R}_{10}}(T) = \sigma_{ap}(T)$, $\sigma_{\mathbf{R}_{11}}(T) = \sigma_{su}(T)$, $\rho_{ap}(T) = \mathbb{C}\backslash\sigma_{ap}(T)$ and $\rho_{su}(T) = \mathbb{C}\backslash\sigma_{su}(T)$. We consider the following classes of bounded linear operators:

$$\mathbf{gDR}_i = \left\{ T \in L(X) : \begin{array}{c} \text{there exists } (M, N) \in Red(T) \text{ such that} \\ T_M \in \mathbf{R}_i \text{ and } T_N \text{ is quasinilpotent} \end{array} \right\}, \quad 1 \leq i \leq 12.$$

If T_N mentioned in this definition is nilpotent then it is said that T belongs to the class $\mathbf{DR}_i$, $1 \leq i \leq 12$. It is clear that $\mathbf{R}_i \subset \mathbf{DR}_i \subset \mathbf{gDR}_i$, $1 \leq i \leq 12$.

We shall say that $T \in L(X)$ is *generalized Drazin upper semi-Fredholm* (resp. *generalized Drazin lower semi-Fredholm, generalized Drazin Fredholm, generalized Drazin upper semi-Weyl, generalized Drazin lower semi-Weyl, generalized Drazin Weyl, generalized Drazin bounded below, generalized Drazin surjective*) if $T \in \mathbf{gD\Phi}_+(X)$ (resp. $\mathbf{gD\Phi}_-(X)$, $\mathbf{gD\Phi}(X)$, $\mathbf{gD}\mathcal{W}_+(X)$, $\mathbf{gD}\mathcal{W}_-(X)$, $\mathbf{gD}\mathcal{W}(X)$, $\mathbf{gD}\mathcal{M}(X)$, $\mathbf{gD}\mathcal{Q}(X)$). The reason for introducing these names is that all these classes generalize the class of generalized Drazin invertible operators and, as we will see, may be characterized in a similar way as the class of generalized Drazin invertible operators. We remark that pseudo B-Fredholm operators and generalized Drazin Fredholm operators coincide, as well as, pseudo B-Weyl operators and generalized Drazin Weyl operators.

The following technical lemma will be useful in the sequel.

Lemma 2.2 *Let $T \in L(X)$ and $(M, N) \in Red(T)$. The following statements hold:*

(i) *$T \in \mathbf{R}_i$ if and only if $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, $1 \leq i \leq 3$ or $7 \leq i \leq 12$, and in that case* $\operatorname{ind}(T) = \operatorname{ind}(T_M) + \operatorname{ind}(T_N)$*;*
(ii) *If $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, then $T \in \mathbf{R}_i$, $4 \leq i \leq 6$;*
(iii) *If $T \in \mathbf{R}_i$ and T_N is Weyl, then $T_M \in \mathbf{R}_i$, $4 \leq i \leq 6$.*

Proof (i): From the equalities $N(T) = N(T_M) \oplus N(T_N)$ and $R(T) = R(T_M) \oplus R(T_N)$ it follows that $\alpha(T) = \alpha(T_M) + \alpha(T_N)$ and $\beta(T) = \beta(T_M) + \beta(T_N)$. It implies that $\alpha(T) < \infty$ if and only if $\alpha(T_M) < \infty$ and $\alpha(T_N) < \infty$, and also, $\beta(T) < \infty$ if and only if $\beta(T_M) < \infty$ and $\beta(T_N) < \infty$. It is known that $R(T)$ is closed if and only if $R(T_M)$ and $R(T_N)$ are closed [13, Lemma 3.3]. Therefore T is bounded below (surjective, upper semi-Fredholm, lower semi-Fredholm) if and only if T_M and T_N are bounded below (surjective, upper semi-Fredholm, lower semi-Fredholm), and in that case $\operatorname{ind}(T) = \alpha(T) - \beta(T) = (\alpha(T_M) + \alpha(T_N)) - (\beta(T_M) + \beta(T_N)) = \operatorname{ind}(T_M) + \operatorname{ind}(T_M)$.

Since $N(T^n) = N(T_M^n) \oplus N(T_N^n)$, for every $n \in \mathbb{N}$, we conclude that $\operatorname{asc}(T) < \infty$ if and only if $\operatorname{asc}(T_M) < \infty$ and $\operatorname{asc}(T_N) < \infty$, with $\operatorname{asc}(T) = \max\{\operatorname{asc}(T_M), \operatorname{asc}(T_N)\}$. Similarly, as $R(T^n) = R(T_M^n) \oplus R(T_N^n)$, $n \in \mathbb{N}$, we get that $\operatorname{dsc}(T) < \infty$ if and only if $\operatorname{dsc}(T_M) < \infty$ and $\operatorname{dsc}(T_N) < \infty$, with $\operatorname{dsc}(T) = \max\{\operatorname{dsc}(T_M), \operatorname{dsc}(T_N)\}$.

(ii): Follows from (i).

(iii): Suppose that $T \in \mathcal{W}_+(X)$ and that T_N is Weyl. According to (i) it follows that $T_M \in \Phi_+(X)$ and $\text{ind}(T_M) = \text{ind}(T_M) + \text{ind}(T_N) = \text{ind}(T) \leq 0$. Thus T_M is upper semi-Weyl. The cases $i = 5$ and $i = 6$ can be proved similarly. □

3 Main Results

We start with the result which is proved in [30] using topological uniform descent.

Lemma 3.1 ([30], Lemma 2.4) *If $T \in L(X)$ admits a GKD(M, N), then there exists a positive constant $\epsilon > 0$, such that*

(i) *$T - \lambda$ is Kato for all $0 < |\lambda| < \epsilon$;*
(ii) *$\alpha(T - \lambda) = \alpha(T_M) \leq \alpha(T)$ for all $0 < |\lambda| < \epsilon$;*
(iii) *$\beta(T - \lambda) = \beta(T_M) \leq \beta(T)$ for all $0 < |\lambda| < \epsilon$.*

It is worth noticing that it can be also derived using the gap theory. Namely, let $T \in L(X)$ admit a GKD(M, N). Then, for every $0 \neq \lambda \in \mathbb{C}$ it holds

$$\alpha(T - \lambda) = \alpha(T_M - \lambda) + \alpha(T_N - \lambda) = \alpha(T_M - \lambda), \tag{3.1}$$

since T_N is quasinilpotent. Also, according to [27, Corollary 12.4], $T_M - \lambda$ is Kato for all λ in a neighborhood of 0. From [27, Theorem 12.2] it follows that $\lim_{\lambda \to 0} \hat{\delta}(N(T_M), N(T_M - \lambda)) = 0$ and hence, there exists $\epsilon > 0$ such that $T_M - \lambda$ is Kato and $\hat{\delta}(N(T_M), N(T_M - \lambda)) < 1$ for all $|\lambda| < \epsilon$. Applying (2.3), for all $|\lambda| < \epsilon$, we obtain $\dim N(T_M - \lambda) = \dim N(T_M)$. Now, we use (3.1) and get the statement (ii) of Lemma 3.1. The statement (iii) can be proved similarly by using the implication (2.5).

The following proposition will be used frequently in this article, but we omit its proof since it easily follows from Lemma 3.1 and [27, Lemma 20.9].

Proposition 3.2 *Let $T \in L(X)$. Then the following implications hold:*

(i) *If T is Kato and $0 \in \text{acc}\, \rho_{\Phi_+}(T)$ ($0 \in \text{acc}\, \rho_{\Phi_-}(T)$), then T is upper (lower) semi-Fredholm;*
(ii) *If T is Kato and $0 \in \text{acc}\, \rho_{\mathcal{W}_+}(T)$ ($0 \in \text{acc}\, \rho_{\mathcal{W}_-}(T)$), then T is upper (lower) semi-Weyl;*
(iii) *If T is Kato and $0 \in \text{acc}\, \rho_{ap}(T)$ ($0 \in \text{acc}\, \rho_{su}(T)$), then T is bounded below (surjective);*
(iv) *If T is Kato and $0 \in \text{acc}\, \rho_{\mathcal{B}_+}(T)$ ($0 \in \text{acc}\, \rho_{\mathcal{B}_-}(T)$), then T is bounded below (surjective).*

Proposition 3.3 *Let $T \in L(X)$ and $1 \leq i \leq 12$. If T belongs to the set* $\mathbf{gDR}_i$, *then* $0 \notin \text{acc}\, \sigma_{\mathbf{R}_i}(T)$.

Proof Let $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Since $\mathbf{R}_i$ is open, there exists $\epsilon > 0$ such that $(T - \lambda)_M = T_M - \lambda \in \mathbf{R}_i$ for $|\lambda| < \epsilon$. On the other hand, $(T - \lambda)_N = T_N - \lambda \in L(X)^{-1} \subset \mathbf{R}_i$ for every $\lambda \neq 0$. Now by applying Lemma 2.2 we obtain that $T - \lambda \in \mathbf{R}_i$ for $0 < |\lambda| < \epsilon$, and so $0 \notin \text{acc}\, \sigma_{\mathbf{R}_i}(T)$. □

We now state the first main result.

Theorem 3.4 *Let $T \in L(X)$ and $1 \le i \le 6$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent, that is $T \in \mathbf{gDR}_i$;*
(ii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$;*
(iii) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$;*
(iv) *There exists a projection $P \in L(X)$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is quasinilpotent.*

Proof (i) $\Longrightarrow$ (ii): Let $T = T_M \oplus T_N$, where $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Then $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$ by Proposition 3.3. From [27, Theorem 16.21] it follows that there exist two closed T-invariant subspaces M_1 and M_2 such that $M = M_1 \oplus M_2$, M_2 is finite dimensional, T_{M_1} is Kato and T_{M_2} is nilpotent. We have $X = M_1 \oplus (M_2 \oplus N)$, $M_2 \oplus N$ is closed, $T_{M_2 \oplus N} = T_{M_2} \oplus T_N$ is quasinilpotent and thus T admits the GKD $(M_1, M_2 \oplus N)$.

(ii) $\Longrightarrow$ (iii): Clear.

(iii) $\Longrightarrow$ (i): Let $i \in \{1, 2, 3\}$. Assume that T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$, that is $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent, and also, because of $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T)$, according to Lemma 2.2(i), it follows that $0 \in \operatorname{acc} \rho_{\mathbf{R}_i}(T_M)$. From Proposition 3.2(i) it follows that $T_M \in \mathbf{R}_i$, and so $T \in \mathbf{gDR}_i$.
Suppose that T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathcal{W}_+}(T)$, i.e. $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$. Then there exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent. We show that $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$. Let $\epsilon > 0$. From $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $T - \lambda \in \mathcal{W}_+(X)$. As T_N is quasinilpotent, $T_N - \lambda$ is invertible, and so, according to Lemma 2.2(iii), we conclude that $T_M - \lambda \in \mathcal{W}_+(M)$, that is $\lambda \in \rho_{\mathcal{W}_+}(T_M)$. Therefore, $0 \in \operatorname{acc} \rho_{\mathcal{W}_+}(T_M)$ and from Proposition 3.2(ii) it follows that T_M is upper semi-Weyl, and so $T \in \mathbf{gD}\mathcal{W}_+(X)$. The cases $i = 5$ and $i = 6$ can be proved similarly.

(i) $\Longrightarrow$ (iv): Suppose that there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. Let $P \in L(X)$ be a projection such that $N(P) = M$ and $R(P) = N$. Then $TP = PT$ and every element $x \in X$ may be represented as $x = x_1 + x_2$, where $x_1 \in M$ and $x_2 \in N$. Also,

$$\|(TP)^n x\|^{\frac{1}{n}} = \|T^n P x\|^{\frac{1}{n}} = \|(T_N)^n x_2\|^{\frac{1}{n}} \to 0 \ (n \to \infty),$$

since T_N is quasinilpotent. We obtain $H_0(TP) = X$, so TP is quasinilpotent. Since $(T + P)_M = T_M$ and $(T + P)_N = T_N + I_N \in L(N)^{-1}$, where I_N is identity on N, we have that $(T + P)_M \in \mathbf{R}_i$ and $(T + P)_N \in \mathbf{R}_i$ and hence, $T + P \in \mathbf{R}_i$ by Lemma 2.2(i) and (ii).

(iv) $\Longrightarrow$ (i): Assume that there exists a projection $P \in L(X)$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is quasinilpotent. Put $N(P) = M$ and $R(P) = N$. Then $X = M \oplus N$, $T(M) \subset M$ and $T(N) \subset N$. For every $x \in N$ we have

$$\|(T_N)^n x\|^{\frac{1}{n}} = \|T^n P^n x\|^{\frac{1}{n}} = \|(TP)^n x\|^{\frac{1}{n}} \to 0 \ (n \to \infty),$$

since TP is quasinilpotent. It follows that $H_0(T_N) = N$ and hence, T_N is quasinilpotent. It remains to prove that $T_M \in \mathbf{R}_i$. For $i \in \{1, 2, 3\}$, by Lemma 2.2(i) we deduce that $T_M = (T + P)_M \in \mathbf{R}_i$. Set $i = 4$. Since T_N is quasinilpotent, it follows that $T_N + I_N$ is invertible, where I_N is identity on N. From $T + P \in \mathcal{W}_+(X)$ and the decomposition

$$T + P = (T + P)_M \oplus (T + P)_N = T_M \oplus (T_N + I_N),$$

according to Lemma 2.2(iii), we conclude that $T_M \in \mathcal{W}_+(M)$. For $i = 5$ and $i = 6$ we apply similar consideration. $\square$

Jiang and Zhong show in [13, Theorem 3.5 and Theorem 3.9] that if $T - \lambda_0 \in L(X)$ admits a GKD, $\sigma_{ap}(T)$ ($\sigma_{su}(T)$) does not cluster at λ_0 if and only if λ_0 is not an interior point of $\sigma_{ap}(T)$ ($\sigma_{su}(T)$). Corollary 3.5 extend this result to the cases of the essential spectra, while in Theorems 3.6 and 3.7 we provide further conditions that are equivalent to those mentioned above.

Corollary 3.5 *Let $T \in L(X)$ and $1 \leq i \leq 6$. If $T - \lambda_0$ admits a generalized Kato decomposition, then $\sigma_{\mathbf{R}_i}(T)$ does not cluster at λ_0 if and only if λ_0 is not an interior point of $\sigma_{\mathbf{R}_i}(T)$.*

Proof Follows from the equivalence (ii)$\Longleftrightarrow$(iii) of Theorem 3.4. $\square$

Theorem 3.6 *Let $T \in L(X)$. The following conditions are equivalent:*

- (i) *$H_0(T)$ is closed and there exists a closed subspace M of X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed;*
- (ii) *There exists $(M, N) \in Red(T)$ such that T_M is bounded below and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{M}(X)$;*
- (iii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{ap}(T)$;*
- (iv) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{ap}(T)$;*
- (v) *There exists a bounded projection P on X which commutes with T such that $T + P$ is bounded below and TP is quasinilpotent;*
- (vi) *There exists $(M, N) \in Red(T)$ such that T_M is upper semi-Browder and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{B}_+(X)$;*
- (vii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_+}(T)$;*
- (viii) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}_+}(T)$;*
- (ix) *There exists a bounded projection P on X which commutes with T such that $T + P$ is upper semi-Browder and TP is quasinilpotent.*

  In particular, if T satisfies any of the conditions (i)–(ix)*, then the subspace N in* (ii) *is uniquely determined and $N = H_0(T)$.*

Proof (i) $\Longrightarrow$ (ii): Suppose that $H_0(T)$ is closed and that there exists a closed T-invariant subspace M of X such that $X = H_0(T) \oplus M$ and $T(M)$ is closed. For $N = H_0(T)$ we have that $(M, N) \in Red(T)$ and $H_0(T_N) = N$, which implies that T_N is quasinilpotent. From $N(T_M) = N(T) \cap M \subset H_0(T) \cap M = \{0\}$ it follows that T_M is injective and since $R(T_M) = T(M)$ is a closed subspace in M, we conclude that T_M is bounded below.

(ii) $\Longrightarrow$ (i): Assume that there exists $(M, N) \in Red(T)$ such that T_M is bounded below and T_N is quasinilpotent. Then (M, N) is a GKD for T, and so from [1, Corollary 1.69] it follows that $H_0(T) = H_0(T_M) \oplus H_0(T_N) = H_0(T_M) \oplus N$. Since T_M is bounded below, we get that $H_0(T_M) = \{0\}$ and hence $H_0(T) = N$. Therefore, $H_0(T)$ is closed and complemented with M, $(M, H_0(T)) \in Red(T)$, and $T(M)$ is closed because T_M is bounded below.
The implications (ii) $\Longrightarrow$ (iii) and (vi) $\Longrightarrow$ (vii) can be proved analogously to the proof of the implication (i) $\Longrightarrow$ (ii) in Theorem 3.4. The implications (iii) $\Longrightarrow$ (iv) and (vii) $\Longrightarrow$ (viii) are clear.

(viii) $\Longrightarrow$ (ii): Let T admit a GKD and let $0 \notin \operatorname{int} \sigma_{\mathcal{B}_+}(T)$, i.e. $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$. There exists $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent. From $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ it follows that $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T_M)$ according to Lemma 2.2(i). From Proposition 3.2(iv) it follows that T_M is bounded below, and hence $T \in \mathbf{gD}\mathcal{M}(X)$.

(iv) $\Longrightarrow$ (ii): This implication can be proved by using Proposition 3.2(iii), analogously to the proof of the implication (viii) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (vi): Follows from the fact that every bounded below operator is upper semi-Browder.

The equivalences (v) $\Longleftrightarrow$ (ii) and (vi) $\Longleftrightarrow$ (ix) can be proved analogously to the equivalence (i) $\Longleftrightarrow$ (iv) in Theorem 3.4. □

Theorem 3.7 *For $T \in L(X)$ the following conditions are equivalent:*

(i) *$K(T)$ is closed and there exists a closed subspace N of X such that $N \subset H_0(T)$ and $(K(T), N) \in Red(T)$;*
(ii) *There exists $(M, N) \in Red(T)$ such that T_M is surjective and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{Q}(X)$;*
(iii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{su}(T)$;*
(iv) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{su}(T)$;*
(v) *There exists a bounded projection P on X which commutes with T such that $T + P$ is surjective and TP is quasinilpotent;*
(vi) *There exists $(M, N) \in Red(T)$ such that T_M is lower semi-Browder and T_N is quasinilpotent, that is $T \in \mathbf{gD}\mathcal{B}_-(X)$;*
(vii) *T admits a GKD and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_-}(T)$;*
(viii) *T admits a GKD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}_-}(T)$;*
(ix) *There exists a bounded projection P on X which commutes with T such that $T + P$ is lower semi-Browder and TP is quasinilpotent.*
In particular, if T satisfies any of the conditions (i)–(ix)*, then the subspace M in* (ii) *is uniquely determined and $M = K(T)$.*

Proof (i) $\Longrightarrow$ (ii): Assume that $K(T)$ is closed and that there exists a closed T-invariant subspace N, such that $N \subset H_0(T)$ and $X = K(T) \oplus N$. For $M = K(T)$ we have that $(M, N) \in Red(T)$, $R(T_M) = R(T) \cap M = R(T) \cap K(T) = K(T) = M$, and so T_M is surjective. Since $H_0(T_N) = H_0(T) \cap N = N$, we conclude that T_N is quasinilpotent.

(ii) $\Longrightarrow$ (i): Suppose that there exists $(M, N) \in Red(T)$ such that T_M is surjective and T_N is quasinilpotent. Then (M, N) is a GKD for T and from [1, Theorem

1.41] we obtain that $K(T) = K(T_M)$. Since T_M is surjective, it follows that $K(T_M) = M$, and so $K(T) = M$ and $K(T)$ is closed. Thus $(K(T), N) \in Red(T)$ and since T_N is quasinilpotent, we have that $N = H_0(T_N) \subset H_0(T)$. The rest of the proof is similar to the proofs of Theorems 3.6 and 3.4. □

Remark 3.8 If T is generalized Drazin invertible, then from Theorem 2.1 and Theorem 3.6 it follows that T is generalized Drazin bounded below and in that case the closed subspace M of X which satisfies the condition (i) in Theorem 3.6, i.e. such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed, is uniquely determined-we show that it must be equal to $K(T)$. In other words, the projection P which satisfies the condition (v) in Theorem 3.6 is uniquely determined-it is equal to the spectral idempotent of T corresponding to the set $\{0\}$.

Indeed, from Theorem 3.6 it follows that $T = T_M \oplus T_{H_0(T)}$, T_M is bounded below and $T_{H_0(T)}$ is quasinilpotent. Since T is generalized Drazin invertible, we have that $0 \notin \operatorname{acc}\sigma(T)$, and hence, $0 \notin \operatorname{acc}\sigma(T_M)$. T_M is Kato since it is bounded below, and so by Proposition 3.2(i) we obtain that T_M is invertible. Since T admits a GKD $(M, H_0(T))$, from [1, Theorem 3.15] it follows that $M = K(T)$. The similar observation can be stated in the context of Theorem 3.7.

In the following theorem we give several necessary and sufficient conditions for $T \in L(X)$ to be generalized Drazin invertible.

Theorem 3.9 *Let $T \in L(X)$. The following conditions are equivalent:*

(i) T is generalized Drazin invertible;
(ii) T admits a GKD and $0 \notin \operatorname{int}\sigma(T)$;
(iii) T admits a GKD and $0 \notin \operatorname{int}\sigma_{\mathcal{B}}(T)$;
(iv) T admits a GKD and $0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T)$;
(v) There exists $(M, N) \in Red(T)$ such that T_M is Browder and T_N is quasinilpotent;
(vi) There exists a bounded projection P on X which commutes with T such that $T + P$ is Browder and TP is quasinilpotent.

Proof Similar to the proof of Theorem 3.6. □

From the equivalences (iii)$\Longleftrightarrow$(iv) in Theorems 3.6, 3.7 and 3.9 it follows that the assertion of Corollary 3.5 holds also for $7 \le i \le 12$.

Remark 3.10 Let $T \in L(X)$ be a Riesz operator with infinite spectrum. The spectrum of T is a sequence converging to 0, $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T)$ and $\sigma_{\mathbf{R}_i}(T) = \{0\}$, $1 \le i \le 9$. It follows that $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T) = \emptyset$, $1 \le i \le 12$, and $0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T) = \emptyset$, $1 \le i \le 9$. On the other hand, it was shown in [13] that T does not admit a GKD. It means that the condition that the operator admits a GKD in the statements (iv), (vii) and (viii) of Theorems 3.6 and 3.7, as well as in the statements (ii), (iii) and (iv) of Theorem 3.9 and also, in the statements (ii) and (iii) of Theorem 3.4, can not be omitted.

The following question is natural.

Question 3.11 Does an operator which does not admit a GKD and such that 0 is not an accumulation point of its approximate point (resp. surjective) spectrum exist? If the answer is affirmative, then it means that the condition that T admits a GKD in the statements (iii) of Theorems 3.6 and 3.7 can not be omitted.

Theorem 3.8 in [13] is equivalent to the following assertion: if 0 is a boundary point of $\sigma(T)$, then T admits a GKD if and only if T is generalized Drazin invertible, that is $T = T_M \oplus T_N$ where $0 \notin \sigma(T_M)$ and T_N is quasinilpotent. The following corollary shows that the previous assertion can be extended to the cases of essential spectra, as well the approximate point and surjective spectrum, in other words, this assertion remains true if we replace the ordinary spectrum by $\sigma_{\mathbf{R}_i}$, $i = 1, \ldots, 11$.

Corollary 3.12 *Let $T \in L(X)$ and let $0 \in \partial\sigma_{\mathbf{R}_i}(T)$, $1 \leq i \leq 12$. Then T admits a generalized Kato decomposition if and only if T belongs to the class $\mathbf{gDR}_i$, that is $T = T_M \oplus T_N$, where $0 \notin \sigma_{\mathbf{R}_i}(T_M)$ and T_N is quainilpotent.*

Proof Follows from the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 3.4, the equivalences (ii)$\Longleftrightarrow$(iv) in Theorems 3.6 and 3.7, the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.9. $\square$

Remark 3.13 From the equivalences (i)$\Longleftrightarrow$(ii) in Theorem 3.4, (ii)$\Longleftrightarrow$(iii) in Theorems 3.6 and 3.7, (i)$\Longleftrightarrow$(iii) in Theorem 2.1, it follows equalities:

$$\begin{aligned}
\mathbf{gD\Phi}(X) &= \mathbf{gD\Phi}_+(X) \cap \mathbf{gD\Phi}_-(X), \\
\mathbf{gD}\mathcal{W}(X) &= \mathbf{gD}\mathcal{W}_+(X) \cap \mathbf{gD}\mathcal{W}_-(X), \\
L(X)^{KD} &= \mathbf{gD}\mathcal{M}(X) \cap \mathbf{gD}\mathcal{Q}(X).
\end{aligned}$$

The inclusions $L(X)^{KD} \subset \mathbf{gD}\mathcal{M}(X)$ and $L(X)^{KD} \subset \mathbf{gD}\mathcal{Q}(X)$ may be strict.

Example 3.14 Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathbb{C}^{\mathbb{N}_0}$ be the linear space of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ℓ_∞, c and c_0 denote the set of bounded, convergent and null sequences. We write $\ell_p = \{x \in \mathbb{C}^{\mathbb{N}_0} : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$. For $n = 0, 1, 2, \ldots$, let $e^{(n)}$ denote the sequences such that $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. The forward and the backward unilateral shifts U and V are linear operators on $\mathbb{C}^{\mathbb{N}_0}$ defined by

$$Ue^{(n)} = e^{(n+1)} \quad \text{and} \quad Ve^{(n+1)} = e^{(n)}, \quad n = 0, 1, 2, \ldots .$$

For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, $U, V \in L(X)$, $VU = I$ and $\sigma(U) = \sigma(V) = \mathbb{D}$, where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Thus, U is bounded below (and thus U is generalized Drazin bounded below) and since $0 \in \operatorname{acc} \sigma(U)$, U is not generalized Drazin invertible. Also, V is surjective (and hence V is generalized Drazin surjective) and V is not generalized Drazin invertible.

The following remark enables us to give another example.

Remark 3.15 Let $T \in L(X)$ and $(M, N) \in Red(T)$ such that T_M is bounded below (resp. surjective) and T_N is finite rank projection. Set $R(T_N) = N_1$ and $N(T_N) = N_2$.

Then $N_1 \oplus N_2 = N$, $\dim N_1 < \infty$, T_{N_1} is identity and T_{N_2} is zero operator. Also, it is easy to see that $X = (M \oplus N_1) \oplus N_2$, $M \oplus N_1$ is closed and that $T_{M\oplus N_1}$ is bounded below (resp. surjective), hence T is generalized Drazin bounded below (resp. generalized Drazin surjective).

Example 3.16 Let X, U and V be as in Example 3.14. Let us introduce an operator $P : X \to X$ as

$$P(x_1, x_2, x_3, \dots) = (x_1, 0, 0, \dots), \quad (x_1, x_2, x_3, \dots) \in X.$$

It is clear that P is bounded linear projection, $\dim R(P) = 1$ and $\sigma(P) = \{0, 1\}$. We consider the operator $T = U \oplus P$. From $\sigma(T) = \sigma(U) \cup \sigma(P) = \mathbb{D}$ we see that T is not generalized Drazin invertible since 0 is an accumulation point of its spectrum. Since U is bounded below, applying Remark 3.15, we obtain that T is generalized Drazin bounded below. Since $\sigma_{ap}(T) = \sigma_{ap}(T) \cup \sigma_{ap}(P) = \partial\mathbb{D} \cup \{0\}$, T is not bounded below.

A similar consideration shows that $V \oplus P$ is generalized Drazin surjective, but not generalized Drazin invertible and not surjective.

We also show that the inclusions $\mathbf{gD}\mathcal{M}(X) \subset \mathbf{gD}\mathcal{W}_+(X)$ and $\mathbf{gD}\mathcal{Q}(X) \subset \mathbf{gD}\mathcal{W}_-(X)$ can be proper.

Example 3.17 Let U and V be as in Example 3.14 and let $T = U \oplus V$. Then, according to Lemma 2.2(i), T is Fredholm and $\text{ind}(T) = \text{ind}(U) + \text{ind}(V) = 0$. Thus T is Weyl and hence, T is generalized Drazin Weyl. Since $\sigma_{ap}(U) = \sigma_{su}(V) = \partial\mathbb{D}$ and $\sigma_{su}(U) = \sigma_{ap}(V) = \mathbb{D}$, it follows that $\sigma_{ap}(T) = \sigma_{ap}(U) \cup \sigma_{ap}(V) = \mathbb{D}$ and $\sigma_{su}(T) = \sigma_{su}(U) \cup \sigma_{su}(V) = \mathbb{D}$. Therefore, $0 \in \text{acc}\, \sigma_{ap}(T)$ and $0 \in \text{acc}\, \sigma_{su}(T)$ and from Theorems 3.6 and 3.7 it follows that T is neither generalized Drazin bounded below nor generalized Drazin surjective.

Remark 3.18 We remark that

$$\begin{aligned}
\Phi_+(X)\backslash\mathcal{W}_+(X) &\subset \mathbf{gD\Phi}_+(X)\backslash\mathbf{gD}\mathcal{W}_+(X),\\
\Phi_-(X)\backslash\mathcal{W}_-(X) &\subset \mathbf{gD\Phi}_-(X)\backslash\mathbf{gD}\mathcal{W}_-(X),\\
\Phi(X)\backslash\mathcal{W}(X) &\subset \mathbf{gD\Phi}(X)\backslash\mathbf{gD}\mathcal{W}(X).
\end{aligned}$$

Indeed, the set $\Phi_+(X)\backslash\mathcal{W}_+(X) = \{T \in \Phi(X) : \text{ind}(T) > 0\}$ is open since the index is locally constant. Hence the set $\sigma_{\mathcal{W}_+}(T)\backslash\sigma_{\Phi_+}(T) = \rho_{\Phi_+}(T)\backslash\rho_{\mathcal{W}_+}(T)$ is open for every $T \in L(X)$. Let $T \in \Phi_+(X)\backslash\mathcal{W}_+(X)$. Then $T \in \mathbf{gD\Phi}_+(X)$ and $0 \in \sigma_{\mathcal{W}_+}(T)\backslash\sigma_{\Phi_+}(T)$. There exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \sigma_{\mathcal{W}_+}(T)\backslash\sigma_{\Phi_+}(T)$. Hence, $0 \in \text{acc}\, \sigma_{\mathcal{W}_+}(T)$ and $T \notin \mathbf{gD}\mathcal{W}_+(X)$ according to Theorem 3.4. Similarly for the remaining inclusions.

The next example shows that the inclusions $\mathbf{gD}\mathcal{W}_+(X) \subset \mathbf{gD\Phi}_+(X)$, $\mathbf{gD}\mathcal{W}_-(X) \subset \mathbf{gD\Phi}_-(X)$ and $\mathbf{gD}\mathcal{W}(X) \subset \mathbf{gD\Phi}(X)$ can be proper.

Example 3.19 Let U and V be as in Example 3.14. The operators U and V are Fredholm, $\text{ind}(U) = -1$ and $\text{ind}(V) = 1$. Therefore, $U \in \Phi_-(X)\backslash\mathcal{W}_-(X)$

and $V \in \Phi_+(X)\backslash\mathcal{W}_+(X)$, and also $U, V \in \Phi(X)\backslash\mathcal{W}(X)$. Hence, according to Remark 3.18, $U \in \mathbf{gD\Phi}_-(X)\backslash\mathbf{gD}\mathcal{W}_-(X)$, $V \in \mathbf{gD\Phi}_+(X)\backslash\mathbf{gD}\mathcal{W}_+(X)$ and $U, V \in \mathbf{gD\Phi}(X)\backslash\mathbf{gD}\mathcal{W}(X)$.

Proposition 3.20 *Let $T \in L(X)$. If $T \in \mathbf{gDR}_i$, then $T^n \in \mathbf{gDR}_i$ for every $n \in \mathbb{N}$, $1 \le i \le 12$.*

Proof Let $T \in \mathbf{gDR}_i$ and $n \in \mathbb{N}$. Then there exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is quasinilpotent. It implies $T^n = T_M^n \oplus T_N^n$, $T_M^n \in \mathbf{R}_i$, $1 \le i \le 12$, and T_N^n is quasinilpotent. Consequently, $T^n \in \mathbf{gDR}_i$. □

In order to prove the opposite implication we need the following consideration. If $T \in L(X)$ and if $f : U \to \mathbb{C}$ is analytic in a neighbourhood $U \supset \sigma(T)$, then $\operatorname{acc}\sigma(f(T)) \subset f(\operatorname{acc}\sigma(T))$; see proof of [9, Theorem 2]. In addition, if f is non-constant on every component of U, then the opposite inclusion is also true, i.e. $\operatorname{acc}\sigma(f(T)) = f(\operatorname{acc}\sigma(T))$ [11, Lemma 2.3.2]. The important moment in their proofs is the fact that $\sigma(T)$ is a compact set and that it satisfies the spectral mapping theorem. We recall that the spectra $\sigma_{\mathbf{R}_i}(T)$, $i \in \{1, 2, 3, 7, 8, 9, 10, 11, 12\}$, are compact and satisfy the spectral mapping theorem, so using a similar method as in the references mentioned above we can conclude that the analogous assertion holds for these types of spectra.

Lemma 3.21 *If p is a nontrivial complex polynomial and if $i \in \{1, 2, 3, 7, 8, 9, 10, 11, 12\}$, then*

$$\operatorname{acc}\sigma_{\mathbf{R}_i}(p(T)) = p(\operatorname{acc}\sigma_{\mathbf{R}_i}(T)). \tag{3.2}$$

Applying formula (3.2) for $p(t) = t^n$, $n \in \mathbb{N}$, gives

$$0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T) \Longleftrightarrow 0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T^n), \quad i \in \{1, 2, 3, 7, 8, 9, 10, 11, 12\}, \; n \in \mathbb{N}. \tag{3.3}$$

Proposition 3.22 *Let $T \in L(X)$ admit a GKD. If $T^n \in \mathbf{gDR}_i$ for some $n \in \mathbb{N}$, then $T \in \mathbf{gDR}_i$, where $i \in \{1, 2, 3, 7, 8, 9, 10, 11, 12\}$.*

Proof Suppose that T admits a GKD and that $T^n \in \mathbf{gDR}_i$ for some $n \in \mathbb{N}$. Then, $0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T^n)$, and also $0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$ according to (3.3). We apply Theorems 3.4, 3.6, 3.7 or 3.9 and obtain $T \in \mathbf{gDR}_i$. □

4 The Classes $\mathbf{DR}_i$

Analysis similar to that in the proof of Theorem 3.4 gives the following result.

Theorem 4.1 *Let $T \in L(X)$ and $1 \le i \le 12$. The following conditions are equivalent:*

(i) There exists $(M, N) \in Red(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is nilpotent, that is $T \in \mathbf{DR}_i$;
(ii) T is of Kato type and $0 \notin \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$;

(iii) T *is of Kato type and* $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$*;*
(iv) *There exists a projection* $P \in L(X)$ *that commutes with* T *such that* $T + P \in \mathbf{R}_i$ *and* TP *is nilpotent.*

Theorem 2.9 in [2] is equivalent to the following assertion: if 0 is a boundary point of $\sigma(T)$, then T is of Kato type if and only if T is Drazin invertible, that is $T = T_M \oplus T_N$ where $0 \notin \sigma(T_M)$ and T_N is nilpotent. The following corollary shows that in the previous assertion the ordinary spectrum can be replaced by $\sigma_{\mathbf{R_i}}$, $i = 1, \ldots, 11$.

Corollary 4.2 *Let* $T \in L(X)$ *and let* $0 \in \partial\sigma_{\mathbf{R}_i}(T)$, $1 \leq i \leq 12$. *Then* T *is of Kato type if and only if* T *belongs to the class* $\mathbf{DR}_i$*, that is* $T = T_M \oplus T_N$*, where* $0 \notin \sigma_{\mathbf{R}_i}(T_M)$ *and* T_N *is nilpotent.*

Proof Follows from the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 4.1. □

Remark 4.3 Using [3, Theorem 2.7] and [5, Lemma 4.1] we see that if $i = 3$ ($i = 6$) then the conditions (i)–(iv) of Theorem 4.1 are equivalent to the fact that T is B-Fredholm (T is B-Weyl), while if $i = 12$ these conditions are equivalent to the fact that T is Drazin invertible.

Similar to the definitions of the B-Fredholm and B-Weyl operators, the classes $\mathbf{BR}_i$ are introduced and studied [4]. In what follows we want to connect the classes $\mathbf{DR}_i$ and $\mathbf{BR}_i$ for other values of i, but some preparation is needed first. For the case of a Hilbert space see [4, Theorem 3.12].

We recall that for every linear operator T acting on a Banach space X and every $n \in \mathbb{N}_0$ the operator $T_n : R(T^n) \to R(T^n)$ is defined as $T_n x = Tx$ for $x \in R(T^n)$. Clearly, T_n is linear operator and $T_0 = T$. Further, let $c'_n(T) = \dim N(T^{n+1})/N(T^n)$ and $c_n(T) = \dim R(T^n)/R(T^{n+1})$. According to [16, Lemmas 1, 2], $c'_n(T) = \dim(N(T) \cap R(T^n))$ and $c_n(T) = \operatorname{codim}(R(T) + N(T^n))$, so the sequences $(c'_n(T))_n$ and $(c_n(T))_n$ are non-increasing. In particular, $c'_0(T) = \alpha(T)$ and $c_0(T) = \beta(T)$. The sequence $((k_n(T))_n$ is given by

$$k_n(T) = \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))$$

and equivalently

$$k_n(T) = \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)).$$

From this it is easily seen that

$$c'_n(T) = k_n(T) + c'_{n+1}(T) \quad \text{and} \quad c_n(T) = k_n(T) + c_{n+1}(T), \tag{4.1}$$

and that an operator $T \in L(X)$ is Kato if and only if $R(T)$ is closed and $k_i(T) = 0$ for all $i \geq 0$.

Remark 4.4 (i) Suppose that X is a Banach space and let $T \in L(X)$. If $(M, N) \in Red(T)$ and if T_N is nilpotent, then the following statements are equivalent.

(a) $\operatorname{asc}(T_n) < \infty$ for every $n \in \mathbb{N}_0$;
(b) $\operatorname{asc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$;
(c) $\operatorname{asc}(T_M) < \infty$. The implication (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c): Let $\operatorname{asc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$. It is evident that $c'_p(T_n) = 0$ for some p. From [4, Lemma 3.1] it follows that $c'_{n+p}(T) = c'_p(T_n) = 0$ and therefore $\operatorname{asc}(T) < \infty$. According to the proof of Lemma 2.2, we get $\operatorname{asc}(T_M) < \infty$.
(c) $\Longrightarrow$ (a): Suppose that $\operatorname{asc}(T_M) < \infty$ and let $n \in \mathbb{N}_0$. Since T_N is nilpotent then $\operatorname{asc}(T_N)$ is finite, and thus $\operatorname{asc}(T) < \infty$ by the proof of Lemma 2.2. There exists $p \geq n$ such that $c'_p(T) = 0$. From [4, Lemma 3.1] it follows $c'_{p-n}(T_n) = c'_p(T) = 0$, and thus $\operatorname{asc}(T_n) < \infty$.

Similarly, if $(M, N) \in Red(T)$ and if T_N is nilpotent, then the following statements are equivalent.

(a) $\operatorname{dsc}(T_n) < \infty$ for every $n \in \mathbb{N}_0$;
(b) $\operatorname{dsc}(T_n) < \infty$ for some $n \in \mathbb{N}_0$;
(c) $\operatorname{dsc}(T_M) < \infty$.

(ii) If T_n is upper (resp. lower) semi-Fredholm for some $n \geq 0$ then $R(T^m)$ is closed, T_m is upper (resp. lower) semi-Fredholm and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for every $m \geq n$ [6].

The following proposition connects the classes $\mathbf{BR}_i$ and $\mathbf{DR}_i$, for $i \in \{1, 2, 4, 5, 7, 8, 10, 11\}$, in the context of a Banach space.

Proposition 4.5 *Let X be a Banach space. If $T \in L(X)$ and $i \in \{1, 2, 4, 5, 7, 8, 10, 11\}$ then the following statements are equivalent.*

(i) *T is of Kato type and $T \in \mathbf{BR}_i$;*
(ii) *$T \in \mathbf{DR}_i$.*

Proof (i) $\Longrightarrow$ (ii): Suppose that T is of Kato type and that $T \in \mathbf{B\Phi}_+(X)$. There exist two closed T-invariant subspaces M and N such that $X = M \oplus N$, T_M is Kato and T_N is nilpotent of degree d. Also, there exists $n \geq 0$ such that $R(T^n)$ is closed and T_n is upper semi-Fredholm. From $c'_n(T) = \dim(N(T) \cap R(T^n)) = \alpha(T_n) < \infty$ and from the fact that $(c'_k(T))_k$ is a non-increasing sequence, there exists $p \geq \max\{d, n\}$ such that $c'_p(T) = c'_{p+1}(T) = \cdots < \infty$. It follows that $T_N^p = 0$, so $c'_p(T_N) = 0$ and thus $c'_p(T_M) = c'_p(T_M) + c'_p(T_N) = c'_p(T) < \infty$. Since $k_j(T_M) = 0$ for each $j \geq 0$ then (4.1) gives $\alpha(T_M) = c'_0(T_M) = c'_p(T_M) < \infty$. Since T_M has closed range, it follows that T_M is upper semi-Fredholm.
In addition, if $T \in \mathbf{B}\mathcal{M}(X)$, then $c'_n(T) = \alpha(T_n) = 0$, so $\alpha(T_M) = c'_p(T_M) = c'_p(T) = 0$, and hence T_M is bounded below. Further, if $T \in \mathbf{B}\mathcal{B}_+(X)$, then T_M is upper semi-Browder by Remark 4.4.
Let $T \in \mathbf{B}\mathcal{W}_+(X)$. It follows that $R(T^p) = R((T_M)^p) \subset M$, $R(T^p)$ is closed and $\operatorname{ind}(T_p) = \operatorname{ind}(T_n) \leq 0$. Since T_M is upper semi-Fredholm, then $\operatorname{ind}(T_M) = \operatorname{ind}((T_M)_p)$, where $(T_M)_p : R((T_M)^p) \to R((T_M)^p)$. It is evident that $T_p = (T_M)_p$, hence $\operatorname{ind}(T_M) = \operatorname{ind}((T_M)_p) = \operatorname{ind}(T_p) = \operatorname{ind}(T_n) \leq 0$, i.e. $T_M \in \mathcal{W}_+(X)$, so $T \in \mathbf{D}\mathcal{W}_+(X)$.
The remaining part can be proved similarly.

(ii) $\Longrightarrow$ (i): Let $T \in \mathbf{D}\mathcal{W}_+(X)$. There exists $(M, N) \in Red(T)$ such that T_M is upper semi-Weyl and T_N is nilpotent. Then $R(T^p)$ is closed and $R(T^p) = R((T_M)^p) \subset M$ for sufficiently large p. From $T_p = (T_M)_p$ we conclude that T_p is upper semi-Fredholm and $\text{ind}(T_p) = \text{ind}((T_M)_p) = \text{ind}(T_M) \leq 0$. It means that T_p is upper semi-Weyl, so $T \in \mathbf{B}\mathcal{W}_+(X)$. Using the similar technique we can prove the remaining part. □

If $T \in L(X)$ is again a Riesz operator with infinite spectrum, then $0 \in \sigma_{gK}(T) \subset \sigma_{Kt}(T)$, so T is not of Kato type. It means that the condition that T is of Kato type can not be omitted from the statement (iii) of Theorem 4.1 for $1 \leq i \leq 12$, as well as from the statement (ii) of Theorem 4.1 if $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The following example ensures that the condition that T is of Kato type in the statement (ii) of Theorem 4.1 can not be omitted if $i \in \{10, 11, 12\}$.

Example 4.6 Let $Q : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the operator defined by

$$Q(\xi_1, \xi_2, \xi_3, \dots) = \left(0, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots\right), \quad (\xi_1, \xi_2, \xi_3, \dots) \in \ell_2(\mathbb{N}).$$

From $\lim_{n\to\infty} ||Q^n||^{\frac{1}{n}} = \lim_{n\to\infty}(\frac{1}{n!})^{\frac{1}{n}} = 0$ we see that Q is quasinilpotent. It follows that 0 is not an accumulation point of the spectrum (resp. approximate point spectrum, surjective spectrum) of Q. Obviously, Q is the limit of finite rank operators F_n, $n \in \mathbb{N}$, given by

$$F_n(\xi_1, \xi_2, \xi_3, \dots) = \left(0, \xi_1, \frac{1}{2}\xi_2, \dots, \frac{1}{n}\xi_n, 0, 0, \dots\right), \quad n \in \mathbb{N},$$

and therefore Q is compact. Since Q^n is compact and $R(Q^n)$ is infinite dimensional, we conclude that $R(Q^n)$ is not closed for every $n \in \mathbb{N}$.

Suppose that Q is of Kato type, i.e. $Q = Q_M \oplus Q_N$ with Q_M Kato and Q_N nilpotent. For sufficiently large n we have that $R(Q^n) = R((Q_M)^n)$ is closed what is not possible. Consequently, Q is not of Kato type.

5 Applications

For $T \in L(X)$ we define the spectra with respect to the sets $\mathbf{gDR}_i$, $1 \leq i \leq 12$, in a classical way,

$$\sigma_{\mathbf{gDR}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathbf{gDR}_i\}, \quad 1 \leq i \leq 12.$$

From Theorems 3.4, 3.6 and 3.7 it follows that

$$\begin{aligned} \sigma_{\mathbf{gDR}_i}(T) &= \sigma_{gK}(T) \cup \operatorname{acc} \sigma_{\mathbf{R}_i}(T) \\ &= \sigma_{gK}(T) \cup \operatorname{int} \sigma_{\mathbf{R}_i}(T), \ 1 \leq i \leq 12. \end{aligned} \tag{5.1}$$

The spectra $\sigma_{\mathbf{DR}_i}(T)$ are defined analogously. According to Theorem 4.1 and Remark 4.3, we have

$$\sigma_{B\Phi}(T) = \sigma_{Kt}(T) \cup \operatorname{int}\sigma_{\Phi}(T) \ \text{ and } \ \sigma_{B\mathcal{W}}(T) = \sigma_{Kt}(T) \cup \operatorname{int}\sigma_{\mathcal{W}}(T). \tag{5.2}$$

Clearly,

$$\begin{array}{ccccccccc}
 & & \sigma_{\mathbf{gD}\Phi_+}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}_+}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{M}}(T) & \subset & \sigma_{ap}(T) \\
 & \rotatebox{45}{$\subset$} & & & & & & \rotatebox{90}{$\subset$} & \\
\sigma_{gK}(T) & \subset & & \sigma_{\mathbf{gD}\Phi}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}}(T) & & \subset & \sigma_{gD}(T) \\
 & \rotatebox{-45}{$\subset$} & & & & & & \rotatebox{-90}{$\subset$} & \\
 & & \sigma_{\mathbf{gD}\Phi_-}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{W}_-}(T) & \subset & \sigma_{\mathbf{gD}\mathcal{Q}}(T) & \subset & \sigma_{su}(T)
\end{array}$$

According to Remark 3.13 we have

$$\begin{aligned}
\sigma_{\mathbf{gD}\Phi}(T) &= \sigma_{\mathbf{gD}\Phi_+}(T) \cup \sigma_{\mathbf{gD}\Phi_-}(T), \\
\sigma_{\mathbf{gD}\mathcal{W}}(T) &= \sigma_{\mathbf{gD}\mathcal{W}_+}(T) \cup \sigma_{\mathbf{gD}\mathcal{W}_-}(T), \\
\sigma_{gD}(T) &= \sigma_{\mathbf{gD}\mathcal{M}}(T) \cup \sigma_{\mathbf{gD}\mathcal{Q}}(T).
\end{aligned}$$

From (5.1) it follows that if $T \in L(X)$ has the property that

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T),$$

then

$$\sigma_{gK}(T) = \sigma_{\mathbf{gDR}_i}(T), \ 1 \leq i \leq 12.$$

Consequently, if $\sigma(T)$ is at most countable or contained in a line, then $\sigma_{gK}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{gD}(T), \ \ 1 \leq i \leq 11$. As examples of operators with the spectrum contained in a line we mention self-adjoint and unitary operators on a Hilbert space. The spectrum of polynomially meromorphic operator [15] is at most countable.

Proposition 5.1 *Let $T \in L(X)$ and $1 \leq i \leq 12$. The following statements hold:*

(i) $\sigma_{\mathbf{gDR}_i}(T) \subset \sigma_{\mathbf{DR}_i}(T) \subset \sigma_{\mathbf{R}_i}(T) \subset \sigma(T)$*;*
(ii) $\sigma_{\mathbf{gDR}_i}(T)$ *is a compact subset of* $\mathbb{C}$*;*
(iii) $\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$ *consists of at most countably many isolated points.*

Proof (i): It is obvious.

(ii): It suffices to prove that $\sigma_{\mathbf{gDR}_i}(T)$ is closed since it is bounded by the part (i). If $\lambda_0 \notin \sigma_{\mathbf{gDR}_i}(T)$, then $T - \lambda_0 \in \mathbf{gDR}_i$ and by Proposition 3.3 there exists $\epsilon > 0$ such that $T - \lambda_0 - \lambda \in \mathbf{R}_i \subset \mathbf{gDR}_i$ for $0 < |\lambda| < \epsilon$. It means that $D(\lambda_0, \epsilon) \subset \mathbb{C} \setminus \sigma_{\mathbf{gDR}_i}(T)$ and we can conclude that $\sigma_{\mathbf{gDR}_i}(T)$ is closed.

(iii): If $\lambda \in \sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$, then $\lambda \in \sigma_{\mathbf{R}_i}(T)$ and $T - \lambda \in \mathbf{gDR}_i$. Applying Proposition 3.3 we obtain that $\lambda \in \operatorname{iso}\sigma_{\mathbf{R}_i}(T)$, and hence $\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{\mathbf{gDR}_i}(T)$ consists of at most countably many isolated points. □

Corollary 5.2 *Let $T \in L(X)$. Then the following inclusions hold:*

(i) $\operatorname{acc}\sigma_{ap}(T)\backslash\operatorname{acc}\sigma_{\mathcal{B}_+}(T)\subset\sigma_{gK}(T)$,
(ii) $\operatorname{acc}\sigma_{su}(T)\backslash\operatorname{acc}\sigma_{\mathcal{B}_-}(T)\subset\sigma_{gK}(T)$,
(iii) $\operatorname{acc}\sigma(T)\backslash\operatorname{acc}\sigma_{\mathcal{B}}(T)\subset\sigma_{gK}(T)$,
(iv) $\operatorname{int}\sigma_{ap}(T)\backslash\operatorname{int}\sigma_{\mathcal{B}_+}(T)\subset\sigma_{gK}(T)$,
(v) $\operatorname{int}\sigma_{su}(T)\backslash\operatorname{int}\sigma_{\mathcal{B}_-}(T)\subset\sigma_{gK}(T)$,
(vi) $\operatorname{int}\sigma(T)\backslash\operatorname{int}\sigma_{\mathcal{B}}(T)\subset\sigma_{gK}(T)$.

Proof Follows from the equivalences (iii) $\Longleftrightarrow$ (vii) and (iv) $\Longleftrightarrow$ (viii) in Theorems 3.6 and 3.7. □

Remark 5.3 Let $T\in L(X)$ be a Riesz operator with infinite spectrum. As we mentioned earlier, T does not admit a GKD [13]. It is interesting to note that the same follows from Corollary 5.2. Namely, $\sigma_{\mathcal{B}}(T)=\{0\}$ and so $0\notin\operatorname{acc}\sigma_{\mathcal{B}}(T)$, while $0\in\operatorname{acc}\sigma(T)$. Therefore, $0\in\operatorname{acc}\sigma(T)\backslash\operatorname{acc}\sigma_{\mathcal{B}}(T)$ and hence $0\in\sigma_{gK}(T)$ by Corollary 5.2.

We give an alternative proof of the inclusion

$$\partial\sigma(T)\cap\operatorname{acc}\sigma(T)\subset\sigma_{gK}(T)$$

from Jiang and Zhong's paper [13] (see Corollary 3.6 and Theorem 3.8) and establish the inclusions of the same type for other spectra.

Theorem 5.4 *Let $T\in L(X)$. Then the following inclusions hold:*

$$\partial\sigma_{\mathbf{R}_i}(T)\cap\operatorname{acc}\sigma_{\mathbf{R}_i}(T)\subset\sigma_{gK}(T),\qquad 1\le i\le 12. \tag{5.3}$$

Moreover,

$$\begin{aligned}\partial\sigma_{\mathcal{B}_+}(T)\cap\operatorname{acc}\sigma_{ap}(T)&\subset\sigma_{gK}(T);\\ \partial\sigma_{\mathcal{B}_-}(T)\cap\operatorname{acc}\sigma_{su}(T)&\subset\sigma_{gK}(T);\\ \partial\sigma_{\mathcal{B}}(T)\cap\operatorname{acc}\sigma(T)&\subset\sigma_{gK}(T).\end{aligned}$$

Proof From Theorems 3.9 and 2.1, the equivalence (iii)$\Longleftrightarrow$(iv) in Theorems 3.6 and 3.7 and the equivalence (ii)$\Longleftrightarrow$(iii) in Theorem 3.4 it foollows that if $T-\lambda\in L(X)$ admits a GKD, then

$$\lambda\in\operatorname{int}\sigma_{\mathbf{R}_i}(T)\Longleftrightarrow\lambda\in\operatorname{acc}\sigma_{\mathbf{R}_i}(T),\qquad 1\le i\le 12.$$

Therefore, we have the inclusions

$$\partial\sigma_{\mathbf{R}_i}(T)\cap\operatorname{acc}\sigma_{\mathbf{R}_i}(T)=\operatorname{acc}\sigma_{\mathbf{R}_i}(T)\backslash\operatorname{int}\sigma_{\mathbf{R}_i}(T)\subset\sigma_{gK}(T),\qquad 1\le i\le 12.$$

Suppose that $\lambda\in\partial\sigma_{\mathcal{B}_+}(T)$ and $T-\lambda$ admits a GKD. Then $\lambda\notin\operatorname{int}\sigma_{\mathcal{B}_+}(T)$ and from the equivalence (viii)$\Longleftrightarrow$(iii) in Theorem 3.6 we get that $\lambda\notin\operatorname{acc}\sigma_{ap}(T)$. Therefore, if $\lambda\in\partial\sigma_{\mathcal{B}_+}(T)\cap\operatorname{acc}\sigma_{ap}(T)$, then $T-\lambda$ does not admit a GKD, i.e. $\lambda\in\sigma_{gK}(T)$. The remaining inclusions can be proved analogously. □

From (5.3) it follows that

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \cup \operatorname{iso}\sigma_{\mathbf{R}_i}(T), \qquad 1 \le i \le 12,$$

which implies that

$$\partial\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{gK}(T) \subset \operatorname{iso}\sigma_{\mathbf{R}_i}(T),$$

and hence $\partial\sigma_{\mathbf{R}_i}(T)\setminus\sigma_{gK}(T)$ consist of at most countably many points.

The *connected hull* of a compact subset K of the complex plane $\mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C}\setminus K$ ([10, Definition 7.10.1]). Given a compact subset K of the plane, a hole of K is a bounded component of $\mathbb{C}\setminus K$, and so a hole of K is a component of $\eta K\setminus K$. Generally ([10, Theorem 7.10.3]), for compact subsets $H, K \subset \mathbb{C}$,

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H. \tag{5.4}$$

If $K \subseteq \mathbb{C}$ is at most countable, then $\mathbb{C}\setminus K$ is connected and hence, $\eta K = K$. Therefore, for compact subsets $H, K \subseteq \mathbb{C}$,

$$\eta K = \eta H \Longrightarrow (H \text{ is at most countable} \Longleftrightarrow K \text{ is at most countable}), \tag{5.5}$$

and in that case $H = K$.

Theorem 5.5 *Let $T \in L(X)$. Then*

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{\mathbf{gD}\mathcal{M}}(T) & \subset & \partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T) & \subset & \partial\sigma_{\mathbf{gD}\Phi_+}(T) & & \\
 & \subset & & & & & & \subset & \\
\partial\sigma_{gD}(T) & \subset & & \partial\sigma_{\mathbf{gD}\mathcal{W}}(T) & \subset & \partial\sigma_{\mathbf{gD}\Phi}(T) & & \subset & \partial\sigma_{gK}(T) \\
 & \subset & & & & & & \subset & \\
 & & \partial\sigma_{\mathbf{gD}\mathcal{Q}}(T) & \subset & \partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T) & \subset & \partial\sigma_{\mathbf{gD}\Phi_-}(T) & &
\end{array}$$

$$\partial\sigma_{\mathbf{gD\Phi}}(T) \subset \partial\sigma_{\mathbf{gD\Phi}_+}(T), \quad \partial\sigma_{\mathbf{gD\Phi}}(T) \subset \partial\sigma_{\mathbf{gD\Phi}_-}(T),$$
$$\partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T), \quad \partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T),$$

and

$$\begin{aligned}
\eta\sigma_{gK}(T) &= \eta\sigma_{\mathbf{gD\Phi}_+}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}_+}(T) = \eta\sigma_{\mathbf{gD}\mathcal{M}}(T) \\
&= \eta\sigma_{\mathbf{gD\Phi}_-}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}_-}(T) = \eta\sigma_{\mathbf{gD}\mathcal{Q}}(T) \\
&= \eta\sigma_{\mathbf{gD\Phi}}(T) = \eta\sigma_{\mathbf{gD}\mathcal{W}}(T) = \eta\sigma_{gD}(T).
\end{aligned} \tag{5.6}$$

Proof According to (5.4) it is sufficient to prove the inclusions

(i) $\partial\sigma_{gD}(T) \subset \sigma_{gK}(T)$; (ii) $\partial\sigma_{\mathbf{gD}\mathcal{M}}(T) \subset \sigma_{gK}(T)$; (iii)$\partial\sigma_{\mathbf{gD}\mathcal{W}_+}(T) \subset \sigma_{gK}(T)$;

(iv) $\partial\sigma_{\mathbf{gD\Phi}_+}(T) \subset \sigma_{gK}(T)$; (v) $\partial\sigma_{\mathbf{gD}\mathcal{Q}}(T) \subset \sigma_{gK}(T)$; (vi) $\partial\sigma_{\mathbf{gD}\mathcal{W}_-}(T) \subset \sigma_{gK}(T)$;

(vii) $\partial\sigma_{\mathbf{gD\Phi}_-}(T) \subset \sigma_{gK}(T)$; (viii) $\partial\sigma_{\mathbf{gD}\mathcal{W}}(T) \subset \sigma_{gK}(T)$; (ix) $\partial\sigma_{\mathbf{gD\Phi}}(T) \subset \sigma_{gK}(T)$.

Suppose that $\lambda_0 \in \partial\sigma_{gD}(T)$. Since $\sigma_{gD}(T)$ is closed, it follows that

$$\lambda_0 \in \sigma_{gD}(T) = \sigma_{gK}(T) \cup \operatorname{int}\sigma(T). \tag{5.7}$$

We prove that

$$\lambda_0 \notin \operatorname{int}\sigma(T). \tag{5.8}$$

Suppose on the contrary that $\lambda_0 \in \operatorname{int}\sigma(T)$. Then there exists an $\epsilon > 0$ such that $D(\lambda_0, \epsilon) \subset \sigma(T)$. This means that $D(\lambda_0, \epsilon) \subset \operatorname{int}\sigma(T)$ and hence, $D(\lambda_0, \epsilon) \subset \sigma_{gD}(T)$, which contradicts the fact that $\lambda_0 \in \partial\sigma_{gD}(T)$. Now from (5.7) and (5.8), it follows that $\lambda_0 \in \sigma_{gK}(T)$.

The inclusions (ii)-(ix) can be proved similarly to the inclusion (i). □

For $A \subset \mathbb{C}$ it holds

$$A \text{ is at most countable} \Longleftrightarrow \operatorname{acc} A \text{ is at most countable.} \tag{5.9}$$

Proposition 5.6 *Let $T \in L(X)$. Then $\sigma(T)$ is at most countable if and only if $\sigma_{gK}(T)$ is at most countable if and only if $\sigma_{\mathbf{gDR}_i}(T)$ is at most countable for arbitrary $i = 1, \dots, 12$, and in that case $\sigma_{gK}(T) = \sigma_{gD}(T) = \sigma_{\mathbf{gDR}_i}(T)$, $i = 1, \dots, 11$.*

In particular, $\sigma(T)$ is a finite set if and only if $\sigma_{gK}(T) = \emptyset$ if and only if $\sigma_{\mathbf{gDR}_i}(T) = \emptyset$ for arbitrary $i = 1, \dots, 12$.

Proof From (5.9) it follows that $\sigma(T)$ is at most countable if and only if $\sigma_{gD}(T) = \operatorname{acc}\sigma(T)$ is at most countable. Now the first assertion follows from (5.5) and (5.6). The second assertion follows from the first one and the fact that $\sigma(T)$ is a finite set if and only if $\sigma_{gD}(T) = \operatorname{acc}\sigma(T) = \emptyset$. □

The Drazin spectrum of $T \in L(X)$ is defined as

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}.$$

Theorem 5.7 *Let $T \in L(X)$. Then*

$$\partial\sigma_D(T) \subset \partial\sigma_{B\mathcal{W}}(T) \subset \partial\sigma_{B\Phi}(T) \subset \partial\sigma_{Kt}(T)$$

and

$$\eta\sigma_{Kt}(T) = \eta\sigma_{B\Phi}(T) = \eta\sigma_{B\mathcal{W}}(T) = \eta\sigma_D(T).$$

Proof From [2, Theorem 2.9] we have $\sigma_D(T) = \sigma_{Kt}(T) \cup \operatorname{int}\sigma(T)$. Now use this equality and (5.2), and proceed similarly as in the proof of Theorem 5.5. $\square$

The generalized Kato resolvent set of $T \in L(X)$ is defined by $\rho_{gK}(T) = \mathbb{C}\setminus\sigma_{gK}(T)$. We give an alternative proof of Theorem 3 in [14] based on Theorem 5.5.

Corollary 5.8 ([14], Theorem 3) *Let $T \in L(X)$ and let $\rho_{gK}(T)$ has only one component. Then*

$$\sigma_{gK}(T) = \sigma_{gD}(T).$$

Proof Since $\rho_{gK}(T)$ has only one component, it follows that $\sigma_{gK}(T)$ has no holes, and so $\sigma_{gK}(T) = \eta\sigma_{gK}(T)$. From (5.6) it follows that $\sigma_{gD}(T) \supset \sigma_{gK}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{gD}(T) \supset \sigma_{gD}(T)$ and hence $\sigma_{gD}(T) = \sigma_{gK}(T)$. $\square$

Theorem 5.9 *Let $T \in L(X)$ and $1 \leq i \leq 12$. If*

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_i}(T), \tag{5.10}$$

then

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{\mathbf{R}_i}(T) \tag{5.11}$$

and

$$\eta\sigma_{\mathbf{R}_i}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{Kt}(T) = \eta\sigma_{eK}(T). \tag{5.12}$$

Proof From $\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$ it follows that $\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$, and so from (5.3) it follows that $\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T)$. (5.12) follows from (5.11) and (5.4). $\square$

Theorem 5.10 *Let $T \in L(X)$ and $1 \leq i \leq 12$. If*

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T), \tag{5.13}$$

then

$$\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{\mathbf{DR}_i}(T) = \sigma_{\mathbf{R}_i}(T). \tag{5.14}$$

Proof Suppose that $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$ and that every $\lambda \in \sigma_{\mathbf{R}_i}(T)$ is not isolated in $\sigma_{\mathbf{R}_i}(T)$. From Theorem 5.9 it follows that

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{\mathbf{R}_i}(T),$$

and so

$$\sigma_{\mathbf{R}_i}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{gDR}_i}(T).$$

The other cases can be proved analogously. $\square$

If $K \subset \mathbb{C}$ is compact, then for $\lambda \in \partial K$ there is equivalence:

$$\lambda \in \operatorname{acc} K \Longleftrightarrow \lambda \in \operatorname{acc} \partial K. \tag{5.15}$$

Corollary 5.11 *Let $T \in L(X)$ be an operator for which $\sigma_{\mathbf{R}_i}(T) = \partial\sigma(T)$, $1 \le i \le 12$, and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then*

$$\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{\mathbf{DR}_i}(T) = \sigma_{\mathbf{R}_i}(T). \tag{5.16}$$

Proof From $\sigma_{\mathbf{R_i}}(T) = \partial\sigma(T)$ it follows that $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$, while from (5.15) it follows that every $\lambda \in \partial\sigma(T)$ is not isolated in $\partial\sigma(T)$, i.e. in $\sigma_{\mathbf{R}_i}(T)$. Therefore, $\sigma_{\mathbf{R_i}}(T) = \partial\sigma_{\mathbf{R_i}}(T) = \operatorname{acc}\sigma_{\mathbf{R_i}}(T)$ and from Theorem 5.10 we get (5.16). □

Example 5.12 Let U and V be as in Example 3.14. Since $\sigma(U) = \sigma(V) = \mathbb{D}$, $\sigma_{ap}(U) = \sigma_{su}(V) = \partial\mathbb{D}$, $\sigma_\Phi(U) = \sigma_{\Phi_+}(U) = \sigma_{\Phi_-}(U) = \partial\mathbb{D}$ and $\sigma_\Phi(V) = \sigma_{\Phi_+}(V) = \sigma_{\Phi_-}(V) = \partial\mathbb{D}$, then the conditions of Theorem 5.10, as well Corollary 5.11, are satisfied for $i = 1, 2, 3, 10, 11$, and hence we get

$$\begin{aligned}\partial\mathbb{D} &= \sigma_{gK}(U) = \sigma_{Kt}(U) = \sigma_K(U) = \sigma_{ap}(U)\\ &= \sigma_{\mathbf{gD}\mathcal{M}}(U) = \sigma_{\mathbf{gD}\mathcal{W}_+}(U) = \sigma_{\mathbf{gD}\Phi_+}(U) = \sigma_{\mathbf{gD}\Phi_-}(U) = \sigma_{\mathbf{gD}\Phi}(U)\\ &= \sigma_{\mathbf{D}\mathcal{M}}(U) = \sigma_{\mathbf{D}\mathcal{W}_+}(U) = \sigma_{\mathbf{D}\boldsymbol{\Phi}_+}(U) = \sigma_{\mathbf{D}\boldsymbol{\Phi}_-}(U) = \sigma_{B\Phi}(U)\end{aligned}$$

and

$$\begin{aligned}\partial\mathbb{D} &= \sigma_{gK}(V) = \sigma_{Kt}(V) = \sigma_K(V) = \sigma_{su}(V)\\ &= \sigma_{\mathbf{gD}\mathcal{Q}}(V) = \sigma_{\mathbf{gD}\mathcal{W}_-}(V) = \sigma_{\mathbf{gD}\boldsymbol{\Phi}_-}(V) = \sigma_{\mathbf{gD}\boldsymbol{\Phi}_+}(V) = \sigma_{\mathbf{gD}\boldsymbol{\Phi}}(V)\\ &= \sigma_{\mathbf{D}\mathcal{Q}}(V) = \sigma_{\mathbf{D}\mathcal{W}_-}(V) = \sigma_{\mathbf{D}\boldsymbol{\Phi}_-}(V) = \sigma_{\mathbf{D}\boldsymbol{\Phi}_+}(V) = \sigma_{B\Phi}(V).\end{aligned}$$

Example 5.13 Let T be the unilateral weighted right shift defined on $\ell_p(\mathbb{N})$, $1 \le p < \infty$, with the weight sequence $(\omega_n)_{n\in\mathbb{N}}$. Then the spectral radius of T, $r(T)$, is equal to $\lim_{n\to\infty}\sup_{k\in\mathbb{N}}(\omega_k\cdots\omega_{k+n-1})^{1/n}$.

If we suppose that $\lim_{n\to\infty}\inf_{k\in\mathbb{N}}(\omega_k\cdots\omega_{k+n-1})^{1/n} = r(T)$, then, according to [23, Proposition 1.6.15], $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$. Thus, $\sigma_{ap}(T) = \partial\sigma_{ap}(T) = \operatorname{acc}\sigma_{ap}(T)$ and from Theorem 5.10 it follows that

$$\begin{aligned}\sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_K(T) &= \sigma_{ap}(U)\\ &= \sigma_{\mathbf{gD}\mathcal{M}}(U) = \sigma_{\mathbf{gD}\mathcal{W}_+}(U) = \sigma_{\mathbf{gD}\boldsymbol{\Phi}_+}(U)\\ &= \sigma_{\mathbf{D}\mathcal{M}}(T) = \sigma_{\mathbf{D}\mathcal{W}_+}(T) = \sigma_{\mathbf{D}\boldsymbol{\Phi}_+}(T)\\ &= \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}.\end{aligned}$$

On the other side, if we suppose that $c(T) = \lim_{n\to\infty}\inf(\omega_1\cdots\omega_n)^{1/n} = 0$, then $\sigma(T) = \overline{D(0, r(T))}$ (see [1, Corollary 3.118]). For $r(T) > 0$, in [13, Example 3.14], it is proved that $\sigma_{gK}(T) = \sigma(T)$. It implies that $\sigma_{Kt}(T) = \sigma_{\mathbf{gDR}_i}(T) = \sigma_{\mathbf{DR}_i}(T) = \sigma(T)$, $1 \le i \le 12$.

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